LETTERE ALLA REDAZIONE

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A

Broken-Symmetry Condition for Fermion Fields.

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The simplest way to obtain a theory with a spontaneous symmetry breakdown (1) is to impose the broken-symmetry condition (2)

$$\langle 0|\varphi_\alpha(x)|0\rangle \neq 0 \,, \tag{1}$$

where $\varphi_\alpha(x)$ is a scalar boson field (either a basic field or a more complicated construct such as $\bar{\psi}\Gamma_\alpha\psi$ for instance) transforming according to an internal-symmetry group which leaves the equations of motion and commutation relations of the theory invariant.

Most known examples (2-6) of spontaneously broken symmetries satisfy condition (1).

In a theory with local commutation relations, where eq. (1) holds, Goldstone's theorem assumes us that zero-mass bosons will be present.

The question naturally arises whether a broken-symmetry condition analogous to (1) might be imposed on a fermion field

$$\langle 0|\psi_\alpha(x)|0\rangle \neq 0 \,. \tag{2}$$

This is of some interest in connection with the possibility of obtaining Goldstone fermions (cf. discussion following ref. (7)), and one immediately realizes that it could only be done at the expense of breaking the explicit Lorentz invariance of the theory. Since the possibility exists that as far as observables are concerned Lorentz invariance would still hold, as for instance in electrodynamics, in radiation gauge and in the

(1) J. GOLDSTONE, A. SALAM and S. WEINBERG: *Phys. Rev.*, **122**, 345 (1961); R. F. STREATER: *Proc. Roy. Soc.*, **287** A, 510 (1965); D. KASTLER, D. W. ROBINSON and J. A. SWIECA: *Comm. Math. Phys.*, **2**, 108 (1965).

(2) J. GOLDSTONE: *Nuovo Cimento*, **19**, 154 (1961).

(3) Y. NAMBU and G. JONA-LASINIO: *Phys. Rev.*, **122**, 345 (1961).

(4) G. S. GURALNIK and C. R. HAGEN: *Nuovo Cimento*, **43** A, 1 (1966).

(5) G. S. GURALNIK, C. R. HAGEN and T. W. B. KIBBLE: *Phys. Rev. Lett.*, **13**, 585 (1964); W. S. HELLMAN and P. ROMAN: *Phys. Rev.*, **143**, 1247 (1964); N. G. DESPHANDE and S. A. BLUDMAN: *Phys. Rev.*, **146**, 1186 (1966).

(6) P. W. HIGGS: *Phys. Lett.*, **12**, 132 (1964); *Phys. Rev.*, **145**, 1156 (1966).

(7) T. W. B. KIBBLE: *Proc. of the 1967 International Conference on Particles and Fields* (New York, 1967).

Bjorken ([8]) and Guralnik and Hagen ([9]) models, possible applications of the broken-symmetry condition (2) are not *a priori* destroyed by the breakdown of explicit Lorentz invariance.

Besides the trouble with Lorentz invariance, which happens with a broken-symmetry condition for any field with spin larger than zero, for fermion fields an additional complication occurs, since (2) will be shown incompatible with a positive definite metric in the Hilbert space.

In fact consider

$$\langle 0|\psi_\alpha^\dagger(\boldsymbol{x})\,\psi_\alpha(\boldsymbol{y})|0\rangle = |\langle 0|\psi_\alpha|0\rangle|^2 + \sum_j |\langle j|\psi_\alpha|0\rangle|^2 \exp[i\boldsymbol{p}_j(\boldsymbol{x}-\boldsymbol{y})] + \int \varrho(\boldsymbol{p}) \exp[i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})]\,\mathrm{d}^3p\,, \tag{3}$$

where $|0\rangle$, $|j\rangle$ are discrete eigenstates of the momentum operator

$$\boldsymbol{P}|0\rangle = 0\,, \qquad \boldsymbol{P}|j\rangle = \boldsymbol{p}_j|j\rangle \tag{4}$$

and the integral describes the contribution from the continuum of states. For $|\boldsymbol{x}-\boldsymbol{y}|\to\infty$ this integral does not contribute because of the Riemann-Lebesgue lemma and using anticommutation relations one finds

$$0 = \lim_{|\boldsymbol{x}-\boldsymbol{y}|\to\infty} \langle 0|[\psi_\alpha^\dagger(\boldsymbol{x}), \psi_\alpha(\boldsymbol{y})]_+|0\rangle = \tag{5}$$

$$= \lim_{|\boldsymbol{x}-\boldsymbol{y}|\to\infty} \left\{2|\langle 0|\psi_\alpha|0\rangle|^2 + \sum_j |\langle j|\psi_\alpha|0\rangle|^2 \exp[i\boldsymbol{p}_j(\boldsymbol{x}-\boldsymbol{y})] + |\langle j|\psi_\alpha^\dagger|0\rangle|^2 \exp[-i\boldsymbol{p}_j(\boldsymbol{x}-\boldsymbol{y})]\right\}.$$

The r.h.s. of eq. (5) can only vanish assimptotically if

$$\langle 0|\psi_\alpha|0\rangle = 0\,, \qquad \langle j|\psi_\alpha|0\rangle = 0\,, \qquad \langle j|\psi_\alpha^\dagger|0\rangle = 0 \tag{6}$$

and thus the broken-symmetry condition (2) cannot hold.

The only escape from (6) is to use an indefinite metric whereupon the coefficients of the exponentials in (5) may no longer be positive definite and cancellations can occur.

It is interesting to note that in one of the models studied in ([4]) a broken-symmetry condition (2) was introduced by the formal construction of a field

$$\psi^\lambda(x) = U(\lambda)\,\psi(x)\,U^{-1}(\lambda)\,, \tag{7}$$

with $U(\lambda)$ formally a unitary operator

$$U(\lambda) = \exp\left[\lambda(-1)^N \int (\psi^\dagger(\boldsymbol{y}) + \psi(\boldsymbol{y}))\,\mathrm{d}^3y\right] \tag{8}$$

and $\psi(x)$ a free zero-mass Dirac field.

([8]) J. D. Bjorken: *Ann. of Phys.*, **24**, 174 (1963).

([9]) G. S. Guralnik and C. R. Hagen: *Formal breakdown of Lorentz invariance in two-dimensional field theories* (preprint).

The transformed field $\psi^\lambda(x)$ formally satisfies the Dirac equation with zero mass and canonical anticommutation relations.

By looking only at the infinitesimal transformations

$$\psi^{\delta\lambda}(x) = \psi(x) + \delta\lambda(-1)^N \tag{9}$$

with

$$\langle 0|\psi^{\delta\lambda}(x)|0\rangle = \delta\lambda \neq 0 \tag{10}$$

one might be led to the erroneous conclusion that a fermion field satisfying eq. (2) may exist in a Hilbert space with positive metric (*).

The finite transformation however, computed at first for a finite quantization volume V, reads

$$\psi^\lambda(x) = \psi(x) + \frac{(-1)^N}{\sqrt{2V}} \sin(\sqrt{2V}\,\lambda) + \int_V (\psi^\dagger(\mathbf{y}) + \psi(\mathbf{y}))\,\mathrm{d}^3y\, \frac{\cos(\sqrt{2V}\,\lambda) - 1}{\sqrt{2V}}, \tag{11}$$

which becomes rather meaningless in the limit of infinite volume and understood as a weak limit leads to the identity transformation

$$\lim_{V\to\infty} \langle\varphi|\psi^\lambda(x)|\chi\rangle = \langle\varphi|\psi(x)|\chi\rangle \tag{12}$$

for $|\varphi\rangle$, $|\chi\rangle$ out of a dense set of states in the Hilbert space. Thus the construction (7) does not lead to the condition (2) in agreement with our result (6) (**).

By relaxing the requirement on the positive-definiteness of the metric one can easily build a free zero-mass field satisfying eq. (2). Define

$$\psi_\alpha^\lambda(x) = \psi_\alpha(x) + \lambda n_\alpha(a+b) \qquad \{n_\alpha^* n_\alpha = 1\}, \tag{13}$$

where $\psi_\alpha(x)$ is the usual zero-mass free Dirac field and a, b annihilation operators satisfying

$$[a, \psi_\alpha]_+ = [a^\dagger, \psi_\alpha]_+ = (a \leftrightarrow b) = [a, b]_+ = [a^\dagger, b^\dagger]_+ = [a, b^\dagger]_+ = 0 \tag{14}$$

and

$$[a, a^\dagger]_+ = 1, \tag{15a}$$

$$[b, b^\dagger]_+ = -1. \tag{15b}$$

The indefinite metric is introduced through relation (15*b*).

Defining the vacuum state $|0\rangle$ as

$$|0\rangle = (|\Omega\rangle + a^\dagger|\Omega\rangle)/\sqrt{\ }\ , \tag{16}$$

where $|\Omega\rangle$ is the no-particle vacuum, we notice that:

(*) I am informed by Dr. C. R. HAGEN that the authors of (4) knew of the necessity of employing an indefinite metric in their construction although it is not mentioned in their paper.

(**) For free fields this is also a consequence of Doplicher's (10) result. The Fock representation is the only representation with a vacuum for a free massless Dirac field (in a Hilbert space with positive metric and spectrum condition).

(10) S. DOPLICHER: *Comm. Math. Phys.*, 3, 228 (1966).

a) $\psi^{\lambda}_{\alpha}(x)$ satisfies the free Dirac equation with zero mass and canonical anti-commutation relations.

b) The vacuum state $|0\rangle$ is an eigenstate of the energy-momentum operator with zero eigenvalue.

c) The vacuum expectation value of $\psi^{\lambda}_{\alpha}(x)$ is not zero

$$\langle 0|\psi^{\lambda}_{\alpha}(x)|0\rangle = \frac{\lambda n_{\alpha}}{2}\,. \tag{17}$$

d) The Hilbert space generated by the application of polynomials in ψ^{λ}_{α} and $\psi^{\dagger\lambda}_{\alpha}$ on the vacuum is

$$\mathscr{H} = \mathscr{H}_{\mathrm{F}} \otimes \mathscr{H}_2 \otimes \widetilde{\mathscr{H}}_2\,, \tag{18}$$

where $\mathscr{H}_{\mathrm{F}}$ is the Fock space, $\mathscr{H}_2$ the usual two-dimensional space and $\widetilde{\mathscr{H}}_2$ a two-dimensional space endowed with indefinite metric whose basis vectors $|\Omega\rangle$ and $b^{\dagger}|\Omega\rangle$ satisfy

$$\langle\Omega|\Omega\rangle = 1\,, \qquad \langle\Omega|bb^{\dagger}|\Omega\rangle = -1\,. \tag{19}$$

e) The algebra of field operators is reducible since there are operators $(-1)^N(a+b)$, $(-1)^N(a^{\dagger}+b^{\dagger})$ which commute with all the field operators and are not multiples of the identity. Hence one has a theory with many vacua ([11]):

$$|\Omega\rangle\,, \qquad a^{\dagger}|\Omega\rangle\,, \qquad b^{\dagger}|\Omega\rangle\,, \qquad a^{\dagger}b^{\dagger}|\Omega\rangle\,.$$

Due to the indefinite metric, it is not possible to reduce out completely the representation so as to have a theory with a unique vacuum and a broken-symmetry condition. This is a general feature as easily seen from eq. (5).

We have thus seen that the broken-symmetry condition for a fermion field (eq. (2)) is possible only at expense of introducing indefinite metric in the Hilbert space. It is clear that the same also holds for a broken-symmetry condition involving an odd number of fermion fields.

In order to have a reasonable theory based on (2) one has to ensure that:

A) Lorentz invariance is maintained as far as observables are concerned although it is broken for the fermion field.

B) The negative-norm states must not manifest themselves in observable quantities.

To satisfy *A*) and *B*) one expects a theory to have an algebra of observables much smaller than the algebra of field operators meaning some sort of a very large gauge group like for instance in quantum electrodynamics.

A trivial example satisfying *A*) and *B*) is the free field studied above with a suitable

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([11]) R. Haag: *Nuovo Cimento*, **25**, 287 (1962).

definition of observables as being generated by

$$\psi_\alpha^\lambda(x) = \lim_{V\to\infty} \int_V \frac{\psi_\alpha^\lambda(\boldsymbol{x}')\,\mathrm{d}^3x}{V} = \psi_\alpha(x)\,.$$

A more complicated example is the model recently studied by GURALNIK and HAGEN [9] of spontaneous breakdown of Lorentz invariance in a two-dimensional theory of fermions interacting with vector mesons.

By working in terms of Green functions the structure of the underlying Hilbert space is not immediately explicit and the authors do not stress the appearance of an indefinite metric.

They obtain a broken symmetry by imposing

$$\langle 0|j^\mu(x)|0\rangle = \xi^\mu \neq 0 \tag{20}$$

(with j^μ a current), which appears as a consequence of choosing a free Green function for the fermion field

$$\widetilde{G}_0(x) = G_0(x) + \gamma\,, \tag{21}$$

with γ a constant matrix and G_0 the usual free Green function for a Dirac field with zero mass (*). Equation (21) implies

$$\langle 0|\psi_\alpha|0\rangle \neq 0\,, \tag{22}$$

which is thus the basic broken-symmetry condition of their model from which (20) is a consequence, implying indefinite metric (**).

The exact solution of the model given in [9] indicates that by an appropriate definition of observables the theory is reduced to one without a broken-symmetry condition, which is then physically irrelevant, as in our previous example.

The problem of obtaining a model with a nontrivial fermion broken-symmetry condition satisfying *A*) and *B*) is an open one, and certainly a more difficult one than the corresponding still not completely solved problem for a boson broken-symmetry condition.

(*) Clearly eq. (20) could be also obtained by leaving unchanged the Green function and making the replacement $j^\mu \to j^\mu + \xi^\mu$ but this would correspond to an external rather than spontaneous symmetry breakdown.

(**) This is the reason for the difference of results obtained in the Thirring model by GURALNIK and HAGEN [9] and LEUTWYLER [12]. The first two use the indefinite metric, the other not.

[12] H. LEUTWYLER: *Helv. Phys. Acta*, 38, 431 (1965).